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On some Basic Aspects of Ternary Reversible and Quantum Computing

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***Abstract*– This paper addresses basic aspects of reversible and quantum computing in the context of ternary systems. Ternary extensions of the Pauli matrices are presented and interactions with the Vilenkin-Chrestenson matrix are disclosed. The realization of extended ternary Toffoli gates under the Barenco et al. type of structure is shown not to be possible without ancillary lines, meanwhile a Sasanian-Wang-Perkowski type of structure leads to a five elementary gates realization. The presence of entanglement in ternary quantum computing is addressed and illustrated with an example.**

Keywords* : *Ternary Pauli matrices; extended Toffoli gates; entanglement

I INTRODUCTION

As a motivation for this research let the following statement from the specific literature be quoted :

« *Systems of three-state particles (qutrits) have been much under discussion recently because they expand the potential for quantum information processing and they have been realized and controlled experimentally. Specific realizations for use in quantum communication protocols include biphotons [R1], time-bin entangled photons [R2], and photons with orbital angular momentum [R3]. Qutrit quantum computation with trapped ions has been described theoretically [R4], while one-qutrit gates have been demonstrated experimentally with deuterons [R5]. Specific advantages of qutrits over qubits include more secure key distributions [R6], the solution of the Byzantine agreement problem [R7], and quantum coin flipping [R8].*» [11]. (See the corresponding references in the original paper.)

In spite of the former remarks, there are relatively few contributions in the literature related to *real* multiple-valued reversible circuits, as compared to contributions on binary reversible circuits. Most, if not all of them, are dedicated to ternary reversible circuits [3, 15, 16, 20]. Some careful basic mathematical studies support these efforts (see e.g. [1, 4, 10, 11]).

Reversible circuits are realized with fan-out free, feedback free, cascades of reversible gates, where reversible gates realize bijections on a given set of p^n p–valued n–tuples. These constraints suggest that the design of reversible circuits is much more difficult than that of classical ("irreversible") digital circuits. This is one important reason why evolutionary approaches for the design of binary and MV reversible circuits are receiving increasing attention (see e.g. [5, 6, 13, 14]).

The present paper has been mainly motivated by the pair of nicely selfcontained contributions [15, 16], (where basic gates, a circuits design algorithm, a non-trivial test circuit, and details of low level realization are discussed), and by some open questions and suggestions found in an unfortunately *unfinished-unpublished* paper of Marek Perkowski and colleagues [20].

The next section reviews some mathematical aspects, Pauli matrices, representation of the basic ternary values in the Bloch sphere, and the role of the Vilenkin-Chrestenson transform. In a following section an analysis is done to determine whether Barenco et al. type of structures [2], and Sasanian-Wang-Perkowski type of structures [22, 24] may be used in ternary reversible computing, with similar advantages as in the binary case. A last section is devoted to introduce the Entanglement phenomenon and its presence in ternary quantum computing. Some conclusions will close the paper.

II ADDRESSING SOME FORMAL ASPECTS

When mathematicians and physicists write on reversible/quantum computation, their main interest is in contributing to a better understanding of complex aspects of quantum mechanics. Very early they introduce the Pauli matrices [19] and the (normalized) Hadamard-Walsh matrix (see e.g. [1, 3, 11, 17] or some text books (e.g. [7], [18]); moreover they may address the issue of realizations (see e.g. [10, 16], as well as the references of the initial quotation). On the other hand, more engineering oriented authors rather start with Feynman, Toffoli and Fredkin "gates" and aim to develop algorithms to design circuits, and hopefully some day, computers .

In the binary case, the basic Pauli matrices [19] are frequently called σ_x, σ_y, σ_z, or simply X, Y, and Z, given as:

$$X_2 = NOT = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \quad (1)$$

(The indices in Eq. (1) have been added to denote the binary case and to distinguish between binary and ternary Pauli matrices, to be discussed below).

It becomes apparent that $i \cdot X_2 Z_2 = Y_2$. Therefore X_2 and Z_2 are considered as *generators* of all Pauli matrices. Finally, notice that both X_2 and Z_2 are traceless square roots of the identity matrix.

In [20] Marek Perkowski stated the question about appropriate extensions of the Pauli matrices to the ternary case. An answer, based on [1, 11] is the following :

Definition. [1]. Let $B_1 = \{ |\varphi_1\rangle, \ldots, |\varphi_n\rangle \}$ and $B_2 = \{ |\psi_1\rangle, \ldots, |\psi_n\rangle \}$ be two orthonormal bases in the n dimensional state space. They are said to be *mutually unbiased bases (MUB)* iff $|\langle \varphi_i | \varphi_j \rangle| = 1/\sqrt{n}$, for every $i, j \in \{1, \ldots, n\}$.

The Definition may canonically be extended to sets of MUBs. MUBs are important, in the context of minimal number of required measurements for quantum state determination. In [1] it was shown that for any prime p, there exist $p + 1$ MUBs in the space $\mathbb{C}^p$. These bases consist of eigenvectors of the unitary operators Z, X, XZ, …, XZ^{p-1}, where X and Z are *generalizations of the Pauli operators* to quantum systems with more than two states [1].

Readers should be warned that matrices X_3, Y_3 and Z_3 used in what follows as ternary Pauli matrices, should not be confused with the matrices introduced in [17], which share the same symbols, but have a totally different meaning. This is just an unfortunate coincidence.

For $p = 3$, in analogy with the binary case, traceless cubic roots of the 3×3 identity matrix are the proper candidates. Let $\omega = exp(2\pi i/3)$ denote the principal complex cube root of unity, where in this case $\omega^2 = \omega^*$. Moreover let $t \in \{0, 1, 2\}$, and the basic operations in this set be modulo 3. Then, after [11] the following Pauli-generators, as generalizations of the rotations σ_x and σ_z, are defined:

$$X_3 = |t+1\rangle\langle t| \text{ or, in matrix form, } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (2)$$

$$Z_3 = |t\rangle\, \underline{\omega}^t \langle t| \text{ or, in matrix form, } \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \quad (3)$$

Recall that both X_3 and Z_3 are cubic roots of the identity matrix, from where it is easy to see that $X_3^2 = X_3^{-1} = X_3^T$, and $Z_3^2 = Z_3^{-1}$. Moreover $Z_3^2 = Z_3^*$. Therefore both X_3 and Z_3 are Hermitean besides being unitary. Finally notice that $X_3 = \sqrt{X_3^{-1}}$ and $Z_3 = \sqrt{Z_3^{-1}}$.

The next two operators are generated:

$$Y_3 := X_3 Z_3 = |t+1\rangle\, \underline{\omega}^t \langle t| = \begin{bmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix} = \omega^2 Z_3 X_3$$

$$V_3 := X_3 Z_3^2 = |t+1\rangle\, \underline{\omega}^{2t} \langle t| = \begin{bmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix} = Y_3^*, \quad (4)$$

which are also cubic roots of the identity matrix. Notice that all these four ternary Pauli matrices have zero trace as in the binary case.

Comparing back with the binary case, notice that X_2 in binary equals the NOT_2 operation, and permutes the two components of a vector; meanwhile in the ternary case, X_3 does not realize the classical involutive symmetric ternary NOT_3 operation (see below, Eq. (9)), but cyclic shifts by two positions the three components of a vector. There is however consistency, since in the binary case the effect of X_2 may also be considered as a cyclic shift (*by* -1), of *the two elements* of the vector. (Notice that in the ternary case, X_3 could also be defined as shifting by one position, as in [1], and this is also consistent with the binary case. This alternative definition will be denoted here X'_3. It is simple to realize that X'_3 is the inverse of X_3, i.e. $X'_3 = X^2_3$.)

In the ternary case, Z_3 is a diagonal matrix with increasing powers of ω. This is also consistent with the binary case since there, ω = -1 and therefore, $Z_2 = diag(1, -1) = diag(\omega^0, \omega^1)$.

Notice that the standard basis $\{|0\rangle, |1\rangle, |2\rangle\}$ is the set of eigenvectors of Z_3. From Theorem 2.3 of [1] follows that the set of the bases each consisting of the eigenvectors of Z_3, X_3, X_3Z_3, $X_3Z_3^2$ forms a set of MUBs.

Taking the above in consideration, in this paper a different "realization" of the states of the standard basis than the one used in [15] has been chosen:

$$|t\rangle = \omega^t, \; t \in \{0, 1, 2\}. \quad (5)$$

Frequently these states are called *pure* states, meanwhile a qutrit [1] in a superposition state is expressed as $|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$, where $|c_0|^2 + |c_1|^2 + |c_2|^2$ adds up to 1, and each of these magnitude squared coefficients represents the probability that the qutrit will reach the corresponding pure state after measurement.

The next extension (also mentioned in [20]) is the use of the (normalized) Vilenkin-Chrestenson matrix (see e.g. Section 2.5.1 of [9]), as the natural generalization of the (normalized) Hadamard-Walsh matrix used in binary quantum computing.

$$CH = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^\star \\ 1 & \omega^\star & \omega \end{bmatrix}. \quad (6)$$

The normalized Vilenkin-Chrestenson matrix is unitary: its adjoint is its inverse. (Since it is symmetric, the adjoint reduces to the complex conjugate).

[1] (The word "qutrit" has been coined to denote "quantum ternary digit" in the same line of "qubit" [23].)

Furthermore, an additional important property of *CH* is the following:

$$CH \cdot CH = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad (7)$$

which induces the permutation of the two bottom elements of a vector. (This is the unitary matrix specifying the "D" gate of [15]).

As in the binary case with the Walsh-Hadamard matrix, the Vilenkin-Chrestenson matrix has the property of changing a pure state into a *superposition* of states:

$$CH\cdot|0\rangle \Rightarrow \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \frac{|0\rangle+|1\rangle+|2\rangle}{\sqrt{3}}$$

$$CH\cdot|1\rangle \Rightarrow \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \Rightarrow \frac{|0\rangle+\omega|1\rangle+\omega^2|2\rangle}{\sqrt{3}}$$

$$CH\cdot|2\rangle \Rightarrow \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix} \Rightarrow \frac{|0\rangle+\omega^2|1\rangle+\omega|2\rangle}{\sqrt{3}} \qquad (8)$$

(It is simple to see that *these* superposition states may be reversed if multiplied by *CH**; there are however superposition states induced by other reasons.)

This suggests that entangled states in ternary quantum computing are also possible. (Section III). Readers interested in a basic mathematical rather than physical explanation of entanglement may refer to Section 3.2 of [21].

The interaction of *CH* and *Z* leads to the following unitary matrices:

$$CH \cdot Z_3 \cdot CH = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = NOT_3, \qquad (9)$$

$$CH \cdot Z_3^* \cdot CH = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = NOT_3 \cdot X_3 , \qquad (10)$$

(From (7) and (9) follows that $CH \cdot CH \cdot NOT_3 = X_3$)

$$CH \cdot Z_3 \cdot CH^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = X_3^T, \qquad (11)$$

$$CH \cdot Z_3^* \cdot CH^* = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = X_3 \qquad (12)$$

Remark 1: $CH\cdot Z_3\cdot CH$ realizes the unitary matrix for the involutive symmetric ternary NOT. This is consistent with the binary case: recall that $H\cdot Z_2\cdot H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Remark 2: $CH\cdot Z_3^*\cdot CH$ realizes the "E" operator, and $CH\cdot Z_3\cdot CH^*$ realizes the "C1" –(X_3, already the "C2")– cyclic inversions of [15].

Remark 3: The set {*CH*, Z_3} and proper interactions, generate all 6 permutations of the elements of vectors of three elements.

The interactions between *CH* and X_3 lead to:

$$CH \cdot X_3 \cdot CH = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{bmatrix} = Z_3 \cdot CH \cdot CH , \qquad (13)$$

$$CH \cdot X_3 \cdot CH^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} = Z_3. \qquad (14)$$

The interactions between *CH* and Y_3 lead to:

$$CH \cdot Y_3 \cdot CH = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \omega & 0 \\ \omega^2 & 0 & 0 \end{bmatrix} = Z_3 \cdot NOT_3, \qquad (15)$$

$$CH \cdot Y_3 \cdot CH^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{bmatrix} = Z_3 \cdot CH \cdot Z_3 \cdot CH^*. \qquad (16)$$

Since $V_3 = Y_3^*$, interactions between *CH* and V_3 do not provide substantially new unitary matrices.

II About Ternary Generalized Toffoli Gates

Toffoli gates or Controlled-Controlled-NOT gates are universal in reversible binary computing, since they realize the NAND of the controlling signals when the target line is driven by the constant 1.

In ternary reversible computing this is no longer the case, since NOT_3 applied to *minimum*, as a naïve generalization of NAND to ternary, is not universal.

Let *U* be a ternary unitary matrix. In what follows, a *CCU* gate will be called "generalized ternary Toffoli gate" or, simply (ternary) Toffoli gate. The question that will be investigated below is whether a Barenco et al. type of structure [2], would allow the realization of a ternary Toffoli gate, which is a 3-qutrit gate, by using only 2-qutrit gates and no ancillary line(s).

With respect to the general "quantum realizability" of ternary Toffoli gates, most –(if not all)- authors refer to *the* paper of A. Muthukrishnan and C.R. Stroud Jr. [17], who proved that this is possible (at least, in the context of the linear ion trap scheme), requiring that all control signals have the value 2 to activate a processing ternary gate.

An important aspect of the Barenco et al. type of structure is the clear separation between the control area and the target channel. Using the idea of *heterogeneous quantum gates* disclosed in [14], it is straight forward to realize an isolated ternary Toffoli gate over a processing unitary matrix *U*, working on qutrits, using *non-embedded*

binary controlling signals and a 2-qu*bit* XOR gate. In the "ternary part", the controlled gates would be specified by complex-valued unitary matrices realizing the $\sqrt{U}$ and its adjoint. For instance if $U = NOT_3$ then,

$$\sqrt{NOT_3} = \begin{bmatrix} \alpha & 0 & \alpha^* \\ 0 & 1 & 0 \\ \alpha^* & 0 & \alpha \end{bmatrix}, \text{ where } \alpha = \frac{1+i}{2}. \qquad (17)$$

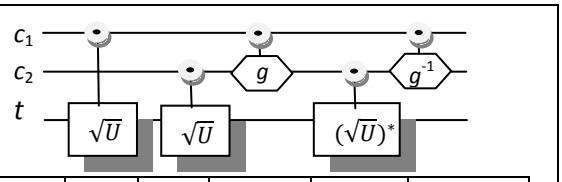
This matrix does not look particularly more complex than

$$\sqrt{NOT_2} = \begin{bmatrix} \beta & \beta^* \\ \beta^* & \beta \end{bmatrix}, \text{ where } \beta = \frac{1+i}{\sqrt{2}}. \qquad (18)$$

However the realization of an *isolated* ternary Toffoli gate does not seem realistic. If circuits are to be constructed, several gates will be used and it is not always possible to avoid that target ternary signals of a gate may be used to control another gate (see e.g. the circuit of [15]). Figure 1 shows the Barenco et al. ternary structure to be analyzed. The basic problem turns out to be the realization of the controlled auxiliary gate *g*, which is irreversible, therefore an ancillary line would be needed and *g* will become a sub-circuit. This contradicts the philosophy of the Barenco et al. structure (of the binary case). A 5-gates Barenco et al. type of realization of a ternary (generalized) Toffoli gate without adding an ancillary line is simply not possible.

The table in Figure 1 shows clearly that the controlled gate *g* needed to generate the control signal for the inverse of $\sqrt{U}$ is not reversible, because of the repeated 2-tuples (shaded cells of the table). An ancillary line would be needed and a sub-circuit would have to replace the intended *g* gate.

A simple circuit to generate the *g* function is shown in Figure 2 where two $CNOT_3$ gates controlled by c_1 and c_2 respectively, (behaving as in [17]), target an ancillary line initialized with 0. It is easy to see that if both control signals are in {0, 1} then, both gates remain inhibited, behaving as the identity, and $g = 0$. If one of them is in {0, 1} and the other has the value 2, then, the corresponding $CNOT_3$ gate will be active and $g = 2$. Finally when both control signals have the value 2, both $CNOT_3$ gates will be active and $g = 0$. This is exactly the needed behaviour according to the table in Figure 1. (To recover the additional 0-signal the same two $CNOT_3$ gates in reverse order may be used.)



c_1 c_2	1st. gate	2nd. gate	Partial transfer	needed	c_1, $g(c_2)$
0,1 0	I_3	I_3	I_3	I_3	0,1 0
0,1 1	I_3	I_3	I_3	I_3	0,1 1
0,1 2	I_3	$\sqrt{U}$	$\sqrt{U}$	$(\sqrt{U})^*$	0,1 2
2 0	$\sqrt{U}$	I_3	$\sqrt{U}$	$(\sqrt{U})^*$	2 2
2 1	$\sqrt{U}$	I_3	$\sqrt{U}$	$(\sqrt{U})^*$	2 2
2 2	$\sqrt{U}$	$\sqrt{U}$	$\boldsymbol{U}$	I_3	2 0,1

Fig. 1: Analysis of the Barenco et al. structure for a ternary Toffoli gate

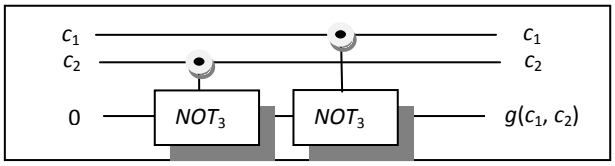
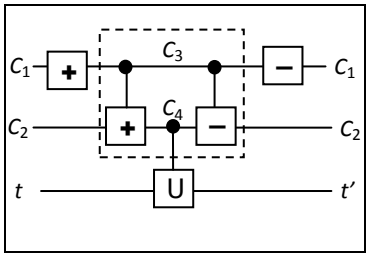


Fig. 2: Reversible sub.-circuit to generate the function *g*.

In her dissertation [22], Zahra Sasanian introduced the *NCV*–$|v_1\rangle$ Library, which comprises gates which are controlled by a complex-valued signal, which is originated by the use of *V* and *CV* gates. The new gates allow very efficient realization of Toffoli gates. A similar result has been reported by Wang and Perkowski [24], however based on a combination of binary and ternary circuits. In analogy to the above works, in the context of a ternary domain a new gate may be specified as follows. Consider a local expansion of the ternary domain into a six-valued domain. Let $\eta = exp(i\pi/3)$. The new domain will be $\{\eta^0, \eta^1, \eta^2, \eta^3, \eta^4, \eta^5\}$, where obviously $\eta^0 = \omega^0 = |0\rangle$, $\eta^2 = \omega^1 = |1\rangle$, $\eta^4 = \omega^2 = |2\rangle$, and η^5 equals ω^2 *rotated by* + $\pi/3$. Rotations are however basic operations in quantum computing. Additionally, consider using Muthukrishnan and Stroud type of controlled gates *in the six-valued domain*, becoming active when driven by the "highest" value: here, η^5, otherwise remaining inhibited and behaving as the identity. Then, the following realization of a *CCU* extended Toffoli gate (with two ternary control signals) by means of 3 (six-valued) controlled-gates on two qudits and 2 non-controlled on .one qudit is possible, as disclosed in Figure 3. The table shows clearly that only when both ternary control signals take the value 2, the internal control signals c_3 and, mainly, c_4 reach the value η^5 needed to activate the *U* gate. The output becomes:

$$t' = (1/4)c_1c_2(c_1\text{-}1)(c_2\text{-}1)\cdot(U - I)\cdot t + I\cdot t$$

or, $\quad t' = U\cdot t$ iff $c_1 = c_2 = 2$.



c_1	c_2	c_3	c_4
0 1	0	η^1 η^3	η^0
	1		η^2
	2		η^4
2	0	$\boldsymbol{\eta^5}$	η^1
	1		η^3
	2		$\boldsymbol{\eta^5}$

Fig. 3: Sasanian-Wang-Perkowski type of realization of extended ternary Toffoli gates.
The dotted box identifies the local six-valued section.
The "+" gates denote rotations by $+\pi/3$. The "–" gates, by $-\pi/3$

III ENTANGLEMENT IN TERNARY QUANTUM COMPUTING

Entanglement is a physical phenomenon that occurs in the quantum context, by which two states develop a special kind of relationship, such that a measurement of one of the states, affects the value of the other. There is no similar effect in the world of classical digital systems.

(The next example follows a similar analysis shown in [18], however adapted to a ternary quantum system.)

Let $|\psi_1\rangle = |\psi_2\rangle = |0\rangle$ and let

$$|\psi_3\rangle = CH\cdot|0\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad (19)$$

$$CH\cdot|0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle).$$

Calculate $|\psi_4\rangle = |\psi_3\rangle \otimes |0\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |10\rangle + |20\rangle)$

$$|\psi_4\rangle = \frac{1}{\sqrt{3}}[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ ]^T. \quad (20)$$

Following the [17] scheme, define

$$CNOT_3 = diag(\mathrm{I}_3, \mathrm{I}_3, NOT_3). \quad (21)$$

Let $|\psi_5\rangle = CNOT_3\cdot(|\psi_3\rangle \otimes |0\rangle)$

$$|\psi_5\rangle = \begin{bmatrix}I_3 & 0_3 & 0_3\\ 0_3 & I_3 & 0_3\\ 0_3 & 0_3 & NOT_3\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}|0\rangle\\|0\rangle\\|0\rangle\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}|0\rangle\\|0\rangle\\|\mathbf{2}\rangle\end{bmatrix}, \quad (22)$$

$$|\psi_5\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |10\rangle + |2\mathbf{2}\rangle) = \frac{1}{\sqrt{3}}[1\ 0\ 0\ 1\ 0\ 0\ \mathbf{0}\ \mathbf{0}\ \mathbf{1}]^{\mathrm{T}}.$$

It is not too difficult to see that there is no pair of *pure* states $|\theta_1\rangle$ and $|\theta_2\rangle$ such that $|\psi_5\rangle = |\theta_1\rangle \otimes |\theta_2\rangle$. It will be said that the original states $|\psi_1\rangle$ and $|\psi_2\rangle$ have reached a *state of entanglement*.

The most interesting effect comes after measurements. Measurements are supported by a projection operator. To measure the probability that a state (in a three-states system) will converge to $|0\rangle$, the following operator will be used:

$$m_0 = |0\rangle\langle 0| = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \quad (23)$$

It is simple to see that m_0 is Hermitean, unitary and idempotent.

If in a 2-qutrit system the probability that the second qutrit will converge to $|0\rangle$ should be calculated, then, an operator $M_0 = I_3 \otimes m_0$ will be used. (If the probability that the first qutrit would converge to $|0\rangle$ should be calculated, then, an operator $m_0 \otimes I_3$ should be used.)

$$M_0 = I_3 \otimes m_0 = diag\,[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ ]. \quad (24)$$

The probability of measuring a 0 at the second qutrit of $|\psi_4\rangle$ is given by:

$$p(0) = \langle\psi_4|M_0|\psi_4\rangle, \quad (25)$$

$$p(0) = \frac{1}{\sqrt{3}}[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ ]\cdot M_0\cdot\frac{1}{\sqrt{3}}[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ ]^T$$

$$= \frac{1}{\sqrt{3}}[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ ]\cdot\frac{1}{\sqrt{3}}[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ ]^T = 1.$$

To obtain $p(0) = 1$ was obvious in this case, due to the definition of $|\psi_4\rangle$ (see Eq. (20)).

The state of the system after measurement is given by:

$$\frac{M_0|\psi_4\rangle}{\sqrt{\langle\psi_4|M_0|\psi_4\rangle}} = \frac{M_0|\psi_4\rangle}{1} \quad (26)$$

and

$$M_0|\psi_4\rangle = \begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\0\\1\\0\\0\\1\\0\\0\end{bmatrix},$$

$$M_0|\psi_4\rangle = \frac{1}{\sqrt{3}}[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0]^T = \frac{1}{\sqrt{3}}\begin{bmatrix}|0\rangle\\|0\rangle\\|0\rangle\end{bmatrix} = |\psi_4\rangle$$

$$= |\psi_3\rangle \otimes |0\rangle = |\psi_3 0\rangle. \quad (27)$$

It becomes fairly clear that the first qutrit remained unchanged, meanwhile the second, after measurement, "with probability 1" converged to $|0\rangle$.

Now the process will be repeated with $|\psi_5\rangle$.

$$p(0) = \langle\psi_5|M_0|\psi_5\rangle, \quad (28)$$

$$p(0) = \frac{1}{\sqrt{3}}[100\ 100\ \mathbf{001}\ ]\cdot M_0\cdot\frac{1}{\sqrt{3}}[100\ 100\ \mathbf{001}\ ]^T$$

$$= \frac{1}{\sqrt{3}}[100\ 100\ \mathbf{000}\ ]\cdot\frac{1}{\sqrt{3}}[100\ 100\ \mathbf{001}\ ]^T = \frac{2}{3}. \quad (29)$$

This result is consistent with Eq. (22).

After the measurement

$$\frac{M_0|\psi_5\rangle}{\sqrt{\langle\psi_5|M_0|\psi_5\rangle}} = \frac{M_0|\psi_5\rangle}{\sqrt{2/3}}, \quad (30)$$

$$\sqrt{\frac{3}{2}}M_0|\psi_5\rangle = \sqrt{\frac{3}{2}}\,(I_3 \otimes |0\rangle\langle 0|)\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}|0\rangle\\|0\rangle\\|\mathbf{2}\rangle\end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}}[1\ 0\ 0\ 1\ 0\ 0\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]^T \Rightarrow \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|0\rangle\,. \quad (31)$$

It is easy to see that after measuring the *second* qutrit, the *first* qutrit changed to a superposition of two pure states. meanwhile the second qutrit converged with probability $\frac{2}{3}$ to $|0\rangle$, (as measured).

Should methods be developed to predict and control entanglement effects, this would open new interesting possibilities in the design of ternary circuits and of quantum inspired algorithms. In [12] it has been shown that « synthesizing entangled circuits and measuring a

reduced number of qu*bits* minimizes the number of operations that must be executed in order to retrieve the result. » No similar result is known to the author for the case of ternary –or in general, MV– quantum circuits. There is in the entanglement related literature however special interest in the ternary version of the Alice-Bob-Eve communication scenario, where qutrits, if not wrong, are not necessarily right (as in the qu*bits* case), but where entanglement does play an important role (see e.g. [25] and its references).

CONCLUSIONS

Basic aspects of quantum computing have been given a consistent formal representation for the ternary case. Ternary versions of the Pauli matrices were disclosed and their consistency with the binary matrices was shown. The Vilenkin-Chrestenson matrix was introduced and its capability to produce superposition of states was illustraded. It was shown that generalized ternary Toffoli gates cannot have a Barenco et al. type of realization unless an ancillary line is included. However a Sasanian-Wang-Perkowski type of realization based on 5 gates on two or one qudits is possible. Entanglement of qutrits was addressed and illustrated by means of a detailled simple example.

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